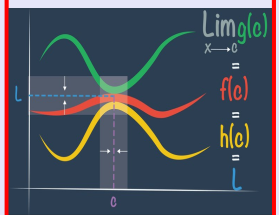
Calculus I

Lecture 6



Feb 19-8:47 AM

Class QZ 5

For $\varepsilon > 0$, find a $\delta > 0$ such that $\lim_{x \to 5}(4x+1) = 21$ ✓

$f(x) = 4x+1$ $\quad a = 5$ $\quad L = 21$ ✓

$\lim_{x \to 5}(4x+1) = 4(5)+1 = 21$ ✓

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|4x+1-21| < \varepsilon$ $\qquad |x-5| < \delta$

$|4x-20| < \varepsilon$

$|4(x-5)| < \varepsilon$

$4|x-5| < \varepsilon$ $\quad \rightarrow \quad |x-5| < \frac{\varepsilon}{4}$

$\boxed{\text{Pick } \delta = \frac{\varepsilon}{4}}$

Jun 25-7:38 AM

Given $f(x) = \frac{1}{x}$

Evaluate $\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{2}{x} + \frac{1}{x-h}}{h^2}$$

LCD $= (x+h)\,x\,(x-h)$

$$= \lim_{h \to 0} \frac{\cancel{(x+h)}x(x-h)\cdot\frac{1}{\cancel{x+h}} - (x+h)\cancel{x}(x-h)\cdot\frac{2}{\cancel{x}} + (x+h)x\cancel{(x-h)}\cdot\frac{1}{\cancel{x-h}}}{h^2(x+h)x(x-h)}$$

$$= \lim_{h \to 0} \frac{x(x-h) - 2\overbrace{(x+h)(x-h)}^{\text{foil}} + x(x+h)}{h^2(x+h)\cdot x(x-h)}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} \cancel{- xh} - 2(\cancel{x^2} - h^2) + \cancel{x^2} \cancel{+ xh}}{h^2(x+h)x(x-h)}$$

$$= \lim_{h \to 0} \frac{2\cancel{h^2}}{\cancel{h^2}(x+h)x(x-h)} = \lim_{h \to 0} \frac{2}{(x+h)x(x-h)}$$

$$= \frac{2}{x \cdot x \cdot x} = \boxed{\frac{2}{x^3}}$$

Jun 25-8:14 AM

Find $f'(x)$ for $f(x) = mx + b$ which is a linear function using $\underbrace{\text{the def. of } f'(x)}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{m(\cancel{x}+h) \cancel{+b} \cancel{-mx} \cancel{-b}}{h} = \lim_{h \to 0} \frac{m\cancel{h}}{\cancel{h}}$$

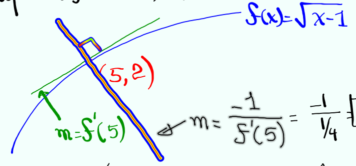
$$= \lim_{h \to 0} m = \boxed{m}$$

$f(x) = mx + b$

$f'(x) = m$

Jun 25-8:23 AM

Find eqn of the normal line to the graph of $f(x)=\sqrt{x-1}$ at $x=5$.



$m = \dfrac{-1}{f'(5)} = \dfrac{-1}{1/4} = \boxed{-4}$

$f(5)=\sqrt{5-1}$
$=\sqrt{4}$
$=2$

Find $f'(x)$ ✓
Evaluate $f'(5)$ → $f'(5) = \dfrac{1}{2\sqrt{5-1}} = \dfrac{1}{4}$
use $y-y_1=m(x-x_1)$, write in slope-Int. Form

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h-1}-\sqrt{x-1}}{h}$$

$$=\lim_{h\to 0}\frac{(\overset{A}{\sqrt{x+h-1}}-\overset{B}{\sqrt{x-1}})(\overset{A}{\sqrt{x+h-1}}+\overset{B}{\sqrt{x-1}})}{h(\sqrt{x+h-1}+\sqrt{x-1})}$$

$$=\lim_{h\to 0}\frac{(\sqrt{x+h-1})^2-(\sqrt{x-1})^2}{h(\sqrt{x+h-1}+\sqrt{x-1})}=\lim_{h\to 0}\frac{\cancel{x}+\overset{1}{\cancel{h}}\cancel{-1}\cancel{-x}\cancel{+1}}{\cancel{h}(\sqrt{x+h-1}+\sqrt{x-1})}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h-1}+\sqrt{x-1}}=\frac{1}{\sqrt{x-1}+\sqrt{x-1}}=\boxed{\frac{1}{2\sqrt{x-1}}}$$

$y-y_1=m(x-x_1)$ $\quad y-2=-4(x-5)$ $\quad \boxed{y=-4x+22}$

Jun 25-8:30 AM

We had the following limits

$$\boxed{\lim_{h\to 0}\frac{\sin h}{h}=1} \quad \text{and} \quad \boxed{\lim_{h\to 0}\frac{1-\cos h}{h}=0}$$

1) Find $\displaystyle\lim_{x\to 0}\frac{\sin 5x}{2x}=\lim_{x\to 0}\frac{1}{2}\cdot\frac{5\sin 5x}{5x}$

$$=\lim_{x\to 0}\left(\frac{5}{2}\cdot\frac{\sin 5x}{5x}\right)$$

$$=\boxed{\lim_{x\to 0}\frac{5}{2}}\cdot\boxed{\lim_{x\to 0}\frac{\sin 5x}{5x}} \to 1$$

$$=\frac{5}{2}\cdot 1=\boxed{\frac{5}{2}}$$

Jun 25-8:49 AM

2) Find $\lim_{x\to 2} \frac{1-\cos(x^2-4)}{x^2-4}$

Let $h = x^2-4$

$x \to 2$

$h \to 2^2-4 \to 0$

$= \lim_{h\to 0} \frac{1-\cos h}{h}$

$= \boxed{0}$

$A^3+B^3 = (A+B)(A^2-AB+B^2)$

$x^3+8 =$

$x^3+2^3 =$

$(x+2)(x^2-2x+4)$

3) Find $\lim_{x\to -2} \frac{\sin(x+2)}{x^3+8}$

$= \lim_{x\to -2} \frac{\sin(x+2)}{(x+2)(x^2-2x+4)}$

$= \lim_{x\to -2} \left[\frac{\sin(x+2)}{x+2} \cdot \frac{1}{x^2-2x+4} \right]$

$= \boxed{\lim_{x\to -2} \frac{\sin(x+2)}{x+2}} \cdot \lim_{x\to -2} \frac{1}{x^2-2x+4}$ $\to 1$

$= 1 \cdot \frac{1}{(-2)^2-2(-2)+4} = 1 \cdot \frac{1}{12} = \boxed{\frac{1}{12}}$

Jun 25-8:59 AM

Find $\lim_{x\to 2} f(x)$

if $x^2-4x+4 \le f(x) \le 4-x^2.$

$\lim_{x\to 2} (x^2-4x+4) = 2^2-4(2)+4 = 0$

$\lim_{x\to 2} (4-x^2) = 4-2^2 = 0$

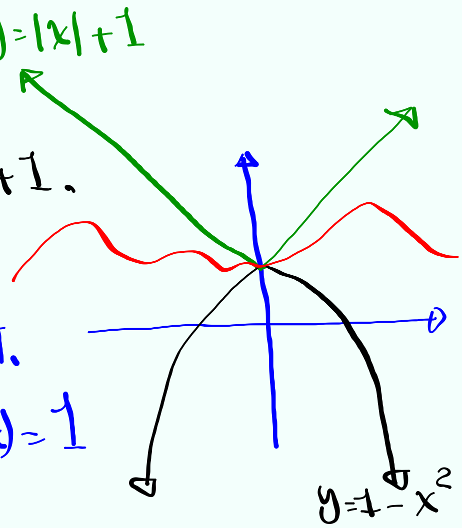
by S.T.,

$\lim_{x\to 2} f(x) = \boxed{0}$

Jun 25-9:11 AM

Find $\lim_{x \to 0} f(x)$

if $1 - x^2 \leq f(x) \leq |x| + 1$.



$\lim_{x \to 0} (1 - x^2) = 1$

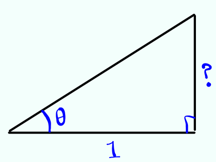
$\lim_{x \to 0} (|x| + 1) = 1$

by S.T.

$\lim_{x \to 0} f(x) = 1$

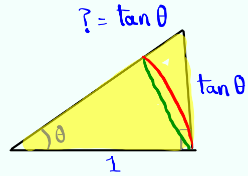
Jun 25-9:15 AM

Consider the right Triangle below:

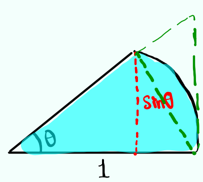


$\tan\theta = \frac{\text{opp.}}{\text{adj.}} = \frac{?}{1}$

$? = \tan\theta$



$\text{Area} = \frac{bh}{2} = \frac{1 \cdot \tan\theta}{2} = \frac{\tan\theta}{2}$

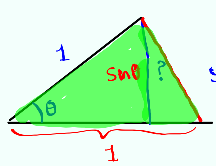


Area of Sector

$A = \frac{1}{2} r^2 \theta$ for $r = 1$

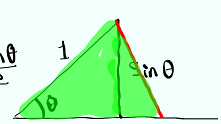
$A = \frac{1}{2}\theta$

Blue < Yellow



$\sin\theta = \frac{\text{opp.}}{\text{hyp.}} = \frac{?}{1}$ $\quad ? = \sin\theta$

$\text{Area} = \frac{bh}{2} = \frac{1 \cdot \sin\theta}{2} = \frac{\sin\theta}{2}$



Jun 25-9:34 AM

Green < Blue < Yellow

$$\frac{\sin\theta}{2} < \frac{1}{2}\theta < \frac{\tan\theta}{2}$$

Multiply by 2

$$\sin\theta < \theta < \tan\theta$$

Divide by $\sin\theta$

$$\frac{\sin\theta}{\sin\theta} < \frac{\theta}{\sin\theta} < \frac{\tan\theta}{\sin\theta}$$

$$\frac{\tan\theta}{\sin\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{\sin\theta}{1}} = \frac{1}{\cos\theta}$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

Take reciprocal

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

$$\lim_{\theta \to 0} 1 = 1$$

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

$$\lim_{\theta \to 0} \cos\theta = \cos 0 = 1$$

By S.T. $\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$

Jun 25-9:49 AM

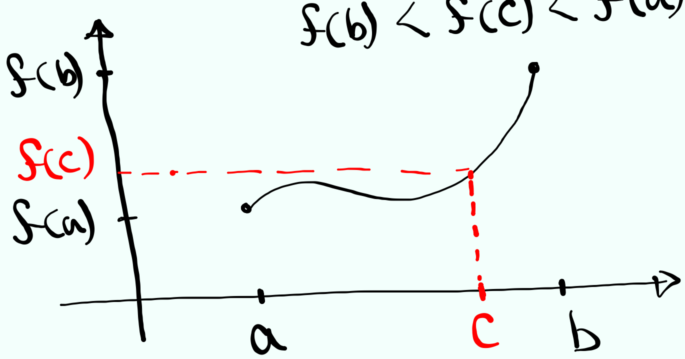
Intermediate Value Theorem:

If $f(x)$ is continous over $[a,b]$,

there is at least one value c in (a,b)

Such that $f(a) < f(c) < f(b)$

$f(b) < f(c) < f(a)$



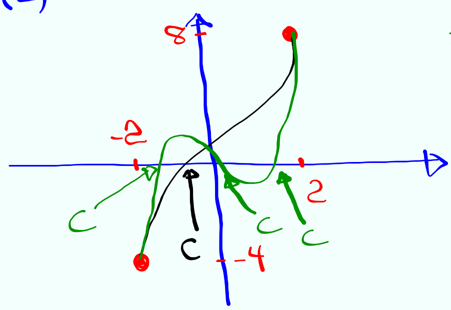
Jun 25-9:57 AM

Show $x^3 - x + 2 = 0$ has a solution between $[-2, 2]$.

$f(x) = x^3 - x + 2$ Polynomial function $\therefore$ Cont. everywhere

$f(-2) = (-2)^3 - (-2) + 2 = -8 + 2 + 2 = -4$

$f(2) = 2^3 - 2 + 2 = 8 - 2 + 2 = 8$



there is at least one value C such that $f(C) = 0$ By I.V.T.

Jun 25-10:02 AM

Show that $x^5 - x^2 + 2x + 3 = 0$ has at least one real solution

$f(x) = x^5 - x^2 + 2x + 3$

Polynomial $\rightarrow$ Continuous $(-\infty, \infty)$

$f(1) = 1^5 - 1^2 + 2(1) + 3 = 5$

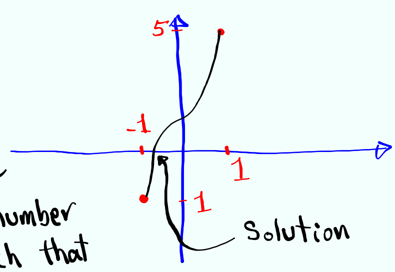
$f(-1) = (-1)^5 - (-1)^2 + 2(-1) + 3 = -1$

$f(-1) < 0 < f(1)$

$-1 < 0 < 5$

By I.V.T., there is at least one number c in $(-1, 1)$ such that

$f(c) = 0$



Jun 25-10:07 AM

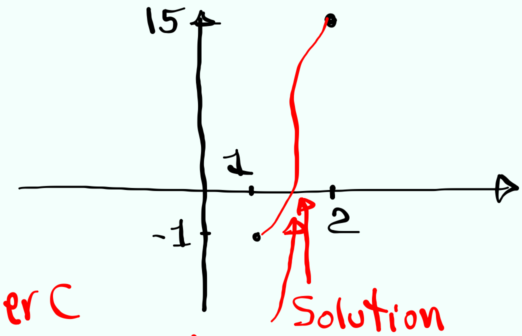
Show $x^4 + x - 3 = 0$ has at least one solution in $(1,2)$.

$f(x) = x^4 + x - 3$

Polynomial $\rightarrow$ Continuous $(-\infty, \infty)$

$f(1) = -1$

$f(2) = 15$



By I.V.T., there is at least one number c in $(1,2)$ such that $f(c) = 0$

Jun 25-10:14 AM

Find c such that

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

is cont. at $x = 2$.

$f(2) = 2^3 - c(2)$
$= 8 - 2c$

$\lim_{x \to 2^-} f(x) = c \cdot 2^2 + 2 \cdot 2 = 4c + 4$

$\lim_{x \to 2^+} f(x) = 2^3 - c \cdot 2 = 8 - 2c$

$\Rightarrow 4c + 4 = 8 - 2c$

$4c + 2c = 8 - 4$

$6c = 4$

$\boxed{c = \frac{2}{3}}$

Jun 25-10:20 AM